

Bouncing Braneworlds

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Abstract

We study cosmological braneworld models with a single *timelike* extra dimension. Such models admit the intriguing possibility that a contracting braneworld experiences a natural bounce without ever reaching a singular state. This feature persists in the case of anisotropic braneworlds under some additional and not very restrictive assumptions. Generalizing our study to braneworld models containing an induced brane curvature term, we find that a FRW-type singularity is once again absent if the bulk extra dimension is timelike. In this case, the universe either has a non-singular origin or commences its expansion from a quasi-singular state during which both the Hubble parameter and the energy density and pressure remain finite while the curvature tensor diverges. The non-singular and quasi-singular behaviour which we have discovered differs both qualitatively and quantitatively from what is usually observed in braneworld models with spacelike extra dimensions and could have interesting cosmological implications.

Following the seminal papers [1,2], most braneworld models with extra dimensions assume that extra dimensions are spacelike, so that the brane is embedded in a Lorentzian multidimensional manifold. However, there is no *a priori* reason why extra dimensions cannot be *timelike*, and the observational constraints on such braneworld models were discussed in [3]. More recently, such models were under consideration in [4–8]. In this letter, we demonstrate that a timelike extra dimension could, in the case of the simplest braneworld model, lead to interesting new features. For instance, a contracting braneworld generically bounces as it reaches a high density thereby leading to the absence of the ‘big crunch’ and ‘big bang’ singularities of general relativity. We consider both Friedmann–Robertson–Walker (FRW) and Bianchi I scenarios and demonstrate that, under some additional and not very restrictive assumptions, the presence of anisotropy does not modify this conclusion.

In this letter, we consider the case where a four-dimensional hypersurface (brane) is the boundary of a five-dimensional Riemannian manifold (bulk) with nondegenerate Lorentzian induced metric. The action of the theory has the natural general form

$$S = M^3 \left[\int_{\text{bulk}} (\mathcal{R} - 2\Lambda) - 2\epsilon \int_{\text{brane}} K \right] + \int_{\text{brane}} (m^2 R - 2\sigma) + \int_{\text{brane}} L(h_{ab}, \phi). \quad (1)$$

Here, \mathcal{R} is the scalar curvature of the five-dimensional metric g_{ab} in the bulk, and R is the scalar curvature of the induced metric $h_{ab} = g_{ab} - \epsilon n_a n_b$ on the brane, where n^a is the vector field of the *inner* unit normal to the brane. The quantity $K = K_{ab} h^{ab}$ is the trace of the symmetric tensor of extrinsic curvature $K_{ab} = h^c_a \nabla_c n_b$ of the brane. The parameter $\epsilon = 1$ if the signature of the bulk space is Lorentzian, so that the extra dimension is spacelike, and $\epsilon = -1$ if the signature is $(-, -, +, +, +)$, so that the extra dimension is timelike. Since the induced metric on the brane is assumed to be Lorentzian, the signature of the extra dimension coincides with the type of the unit normal n^a to the brane. The symbol $L(h_{ab}, \phi)$ denotes the Lagrangian density of the four-dimensional matter fields ϕ the dynamics of which is restricted to the brane so that they interact only with the induced metric h_{ab} . All integrations over the bulk and brane are taken with the natural volume elements $\sqrt{-\epsilon g} d^5x$ and $\sqrt{-h} d^4x$, respectively, where g and h are the determinants of the matrices of components of the metric in the bulk and of the induced metric on the brane, respectively, in the coordinate basis. The symbols M and m denote, respectively, the five-dimensional and four-dimensional Planck masses, Λ is the five-dimensional cosmological constant, and σ is the brane tension. In this paper, we use the notation and conventions of [9].

Variation of action (1) gives the equation of motion in the five-dimensional bulk:

$$\mathcal{G}_{ab} + \Lambda g_{ab} = 0, \quad (2)$$

and on the brane:

$$m^2 G_{ab} + \sigma h_{ab} = \epsilon M^3 S_{ab} + \tau_{ab}, \quad (3)$$

where \mathcal{G}_{ab} and G_{ab} are the Einstein’s tensors of the corresponding spaces, $S_{ab} \equiv K_{ab} - K h_{ab}$, and τ_{ab} denotes the four-dimensional stress–energy tensor of matter on the brane.

Using the Gauss relation on the brane, one obtains the constraint equation

$$\epsilon (R - 2\Lambda) + K_{ab} K^{ab} - K^2 = 0. \quad (4)$$

Then, expressing the extrinsic curvature K_{ab} from (3), one obtains the following scalar equation on the brane [7]:

$$\epsilon M^6 (R - 2\Lambda) + (m^2 G_{ab} + \sigma h_{ab} - \tau_{ab}) (m^2 G^{ab} + \sigma h^{ab} - \tau^{ab}) - \frac{1}{3} (m^2 R - 4\sigma + \tau)^2 = 0, \quad (5)$$

where $\tau = h^{ab}\tau_{ab}$.

Before proceeding further, we pose to discuss the significance of the timelike character of the extra dimension. We emphasize that, in the theory under consideration, the extra dimension is accessible only for the gravitational field, while all matter fields are constrained to the brane and are propagating on the background of the induced Lorentzian metric. Thus, the dynamics of the matter fields on the brane has the standard properties of quantum field theory in curved Lorentzian spacetime, and the effect of the extra timelike dimension tells only in the gravitational sector. An important issue demanding further investigation is related to the tachyonic nature of the Kaluza–Klein gravitational modes that could in principle lead to violation of causality and unitarity on the brane. Some discussion of this issue within the context of braneworld models with more than one time like dimension can be found in [3,4].

In passing, we mention that a generalization of the Randall–Sundrum solution [2] to the case of arbitrary signature of the extra dimension and arbitrary Ricci-flat vacuum brane can be written in the form [4,7]

$$ds^2 = \epsilon dy^2 + \exp\left(-\frac{\epsilon\sigma}{3M^3}y\right) h_{\alpha\beta}(x) dx^\alpha dx^\beta, \quad (6)$$

where $h_{\alpha\beta}(x)$ are the components of the Ricci-flat metric on the brane, which is situated at $y = 0$, and the bulk coordinate is in the range $y \geq 0$. Equation (6) is supplemented by the constraint $\Lambda_{\text{eff}} = 0$, where the expression for Λ_{eff} is given by Eq. (10) below. One can see that, for $\epsilon = -1$ and negative brane tension σ , the gravity appears to be ‘localized’ at the brane (it is, perhaps, more appropriate to speak of ‘deflation’ in the case of timelike extra dimension).

In this letter, we study cosmological implications of this braneworld theory. We first consider a homogeneous and isotropic cosmological model with the cosmological time t , scale factor $a(t)$, energy density $\rho(t)$, and pressure $p(t)$. In this case, by integrating Eq. (5), one obtains the following equation [7]:

$$m^4 \left(H^2 + \frac{\kappa}{a^2} - \frac{\rho + \sigma}{3m^2} \right)^2 = \epsilon M^6 \left(H^2 + \frac{\kappa}{a^2} - \frac{\Lambda}{6} - \frac{C}{a^4} \right), \quad (7)$$

where C is the integration constant, $H \equiv \dot{a}/a$, and $\kappa = 0, \pm 1$ corresponds to the spatial curvature of the universe. For $m = 0$, which corresponds to the Randall–Sundrum limit [2], this equation reduces to

$$H^2 + \frac{\kappa}{a^2} = \frac{\Lambda}{6} + \frac{\epsilon\sigma^2}{9M^6} + \frac{2\epsilon\sigma\rho}{M^6} + \frac{\epsilon\rho^2}{M^6} + \frac{C}{a^4}. \quad (8)$$

Introducing the notation

$$G_N = \frac{3\epsilon\sigma}{4\pi M^6}, \quad (9)$$

$$\Lambda_{\text{eff}} = \frac{\Lambda}{2} + \frac{\epsilon\sigma^2}{3M^6}, \quad (10)$$

one can write Eq. (8) as follows:

$$H^2 + \frac{\kappa}{a^2} = \frac{\Lambda_{\text{eff}}}{3} + \frac{8\pi G_N \rho}{3} + \frac{\epsilon\rho^2}{M^6} + \frac{C}{a^4}. \quad (11)$$

Thus, G_N is the effective gravitational constant, and Λ_{eff} is the effective cosmological constant on the brane. (For $\epsilon = 1$, Eq. (11) reduces to the well-known results [10,11].)

A FRW universe described by (11) is embedded in the bulk with the metric

$$ds^2 = -f(r)du^2 + \frac{\epsilon dr^2}{f(r)} + r^2 d\Omega_\kappa, \quad (12)$$

where

$$f(r) = \epsilon \left(\kappa - \frac{\Lambda r^2}{6} - \frac{C}{r^2} \right) \quad (13)$$

and $d\Omega_\kappa$ is the standard Euclidean three-dimensional metric corresponding to κ , which satisfies the vacuum bulk equation (2). The position of the brane is described by the equation $r = a(u)$.

The function $f(r)$ is assumed to be positive in the domain of action of the coordinate r , which, in particular, implies that at least one of the constants κ , Λ or C must be nonzero. Metric (12) has singularities and horizons in the bulk for certain values of the constants. We will not concern ourselves with this issue, assuming that such singularities can be avoided, for instance, by the introduction of another brane. Although the presence of another brane can affect the evolution and spectrum of perturbations (as in the case in the Randall–Sundrum model), it does not modify the general equations of the cosmological evolution of the brane.

We first discuss some interesting consequences of Eq. (8) or (11) for $\epsilon = -1$. Specifically, as we briefly noted in [8], the fact that the ρ^2 term on the right-hand side of (11) has a negative sign, leads a contracting universe to bounce instead of reaching the singular state $\rho \rightarrow \infty$, $R_{abcd}R^{abcd} \rightarrow \infty$ — typical of general relativistic ‘big crunch’ singularities. In order that the subsequent evolution of the universe be compatible with observations, one requires the brane tension σ to be negative so that the effective gravitational constant, given by (9), is positive. The additional requirement that the bounce take place at densities greater than that during cosmological nucleosynthesis leads to the constraint [11] $|\sigma| \gtrsim (1\text{MeV})^4$. Since $\epsilon = -1$, a braneworld with $\Lambda_{\text{eff}} \geq 0$ in (10), must have a positive bulk cosmological constant $\Lambda > 0$.

As the universe collapses to high densities, the negative ρ^2 term grows much faster than the remaining terms on the right-hand side of (11) leading to $H \rightarrow 0$ and to the inevitable bounce of the braneworld. This feature appears to be quite generic, since it depends neither upon the equation of state of matter nor upon the spatial curvature of the universe. The

simplest singularity-free bouncing braneworld model (with C , κ , and Λ_{eff} equal to zero) has the form

$$H^2 = \frac{8\pi G_N \rho}{3} - \frac{\rho^2}{M^6}. \quad (14)$$

An example of a bouncing universe containing radiation is shown in Fig. 1. It is easy to show that a spatially closed universe ($\kappa = 1$) with matter satisfying $\rho + 3p > 0$ will be ‘cyclic’ in the sense that it will pass through an infinite number of nonsingular expanding-collapsing epochs. As demonstrated in [12], a massive scalar field in such a universe usually leads to an increase in the amplitude of consecutive expansion cycles and to a gradual amelioration of the flatness problem.

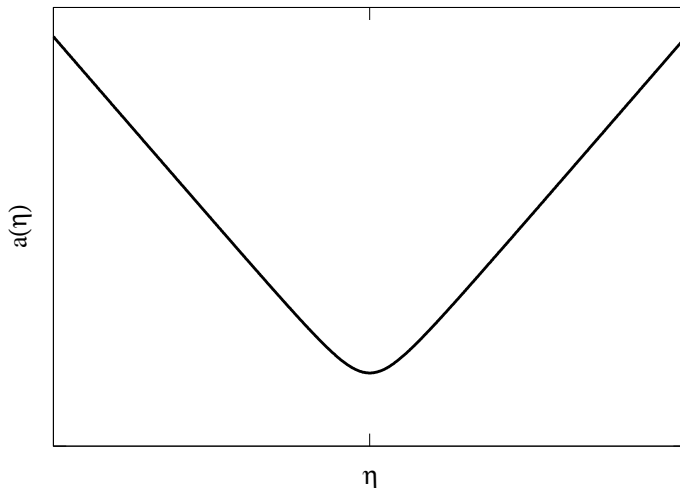


FIG. 1. A bouncing radiation-dominated braneworld ($\eta = \int dt/a$ is the conformal time).

In passing, we note that cosmological braneworld equations with ρ^2 correction terms of negative sign on the right-hand side were also recently discussed in [13] in the context of the Randall–Sundrum two-brane model with a bulk scalar field stabilizing the radion. We also note that bouncing and cyclic braneworlds were discussed in [14] in the context of the usual theory with spacelike extra dimension where the bulk metric is that of a charged anti de Sitter black hole. In this case, the bounce occurs if the black-hole charge is sufficiently high. In our model, the bounce is quite a generic feature.

Next, let us consider the case of a homogeneous but anisotropic braneworld described by the Bianchi I metric

$$ds^2 = -dt^2 + \sum_{i=1}^3 a_i^2(t) (dx^i)^2. \quad (15)$$

In this case, Eq. (5) with $m = 0$ gives the following closed equation on the brane:

$$6\dot{H} + 12H^2 + \sum_{i=1}^3 (H_i - H)^2 = 4\Lambda_{\text{eff}} + \frac{2\epsilon}{3M^6} [\sigma(\rho - 3p) - \rho(\rho + 3p)], \quad (16)$$

where we used the notation of (10) and

$$H_i = \frac{\dot{a}_i}{a_i}, \quad H = \frac{1}{3} \sum_{i=1}^3 H_i = \frac{\dot{a}}{a}, \quad a = (a_1 a_2 a_3)^{1/3}. \quad (17)$$

The last term on the left-hand side of Eq. (16) is the shear scalar of the spatial section of the universe: $\sum_{i=1}^3 (H_i - H)^2 \equiv \sigma_{\alpha\beta} \sigma^{\alpha\beta}$. Because of the presence of this term, and because the evolution of the shear tensor $\sigma_{\alpha\beta}$ is not specified on the brane (see [15]), Eq. (16) cannot be integrated completely, but the result of its integration can be written in the form

$$H^2 + \frac{1}{3a^4} \int \sum_{i=1}^3 (H_i - H)^2 a^3 \dot{a} dt = \frac{\Lambda_{\text{eff}}}{3} + \frac{8\pi G_N \rho}{3} + \frac{\epsilon \rho^2}{M^6} + \frac{C}{a^4}, \quad (18)$$

where the notation is the same as in (9) and (10). We emphasize that Eq. (18) describing the evolution of the Bianchi I brane is absolutely general.

As we noted, the behaviour of the second term on the left-hand side is not specified and, in principle, can be arbitrary. Nevertheless, our conclusion about the bounce of the contracting universe in the case of a timelike extra dimension ($\epsilon = -1$) remains valid as long as the shear scalar $\sigma_{\alpha\beta} \sigma^{\alpha\beta} \equiv \sum_{i=1}^3 (H_i - H)^2$ does not grow faster than a^{-8} as $a \rightarrow 0$ during the contraction of a radiation dominated universe. As before, the bounce is caused by the negative ρ^2 term on the right-hand side. However, in the case of the anisotropic model, the bounce is taking place only as regards the overall expansion \dot{a}/a , while the behaviour of the shear tensor remains unspecified in general; this leaves open the possibility that such a universe bounces only in a given spatial patch.

As an example of bouncing behaviour, the shear can be specified [15,16] by the additional assumption that the so-called nonlocal energy density on the brane \mathcal{U} , which is determined by the projection of the Weyl tensor to the brane, either vanishes or is negligible. This leads to [15]

$$\sigma_{\alpha\beta} \sigma^{\alpha\beta} \equiv \sum_{i=1}^3 (H_i - H)^2 = \frac{6\Sigma^2}{a^6}, \quad \Sigma = \text{const}, \quad (19)$$

which clearly reinforces our earlier conclusion about the bounce. Indeed, the integral in Eq. (18) can now be evaluated to give

$$H^2 = \frac{\Lambda_{\text{eff}}}{3} + \frac{8\pi G_N \rho}{3} + \frac{\epsilon \rho^2}{M^6} + \frac{C}{a^4} + \frac{\Sigma^2}{a^6}, \quad (20)$$

and it is immediately clear that the ρ^2 term on the right-hand side, which grows as a^{-8} during the radiation-dominated contraction stage, dominates over the last (shear) term. The same result is obtained in a slightly more general setup [17], if the nonlocal energy density \mathcal{U} is assumed to behave like radiation, $\mathcal{U} \propto a^{-4}$. Note that, although the universe bounces, the behaviour of the individual components of the shear tensor $\sigma_{\alpha\beta}$ (hence also of the individual components H_i , $i = 1, 2, 3$) is not fully specified in the above examples.

The collapse of a scalar-field dominated universe is even more likely to lead to a bounce because the energy density ρ_ϕ of the field ϕ during contraction becomes kinetic-dominated,

$\dot{\phi}^2/2 \gg V(\phi)$, so that $\rho_\phi \propto a^{-6}$, which makes the ρ^2 term in (11) grow much faster than in the case of radiation [12].

So far, we have been assuming $m = 0$ in the action (1). Let us return to a homogeneous and isotropic universe but drop this constraint, i.e., extend our study to braneworld models whose action contains the induced curvature term on the brane. The cosmological equation (7) corresponding to this model can be solved with respect to the Hubble parameter:

$$\begin{aligned} H^2 + \frac{\kappa}{a^2} &= \frac{\rho + \sigma}{3m^2} + \frac{2\epsilon}{\ell^2} \left[1 \pm \sqrt{1 + \epsilon\ell^2 \left(\frac{\rho + \sigma}{3m^2} - \frac{\Lambda}{6} - \frac{C}{a^4} \right)} \right] \\ &= \frac{\Lambda}{6} + \frac{C}{a^4} + \frac{\epsilon}{\ell^2} \left[1 \pm \sqrt{1 + \epsilon\ell^2 \left(\frac{\rho + \sigma}{3m^2} - \frac{\Lambda}{6} - \frac{C}{a^4} \right)} \right]^2, \end{aligned} \quad (21)$$

where we introduce the length parameter

$$\ell = \frac{2m^2}{M^3}. \quad (22)$$

Similar equations were obtained in [18] for the case of $\epsilon = 1$. The ‘ \pm ’ signs in (21) correspond to two different ways of bounding the bulk space by the brane, depending on whether the inner normal to the brane points in the direction of increasing or decreasing bulk coordinate r in (12). Alternatively, the two different signs in (21) could correspond to the two possible signs of the five-dimensional Planck mass M . The model with the ‘ $-$ ’ sign is the one that passes smoothly to the previously considered case of $m = 0$ in the sense that the right-hand side of (21) with ‘ $-$ ’ sign can formally be expanded in powers of m . The right-hand side of (21) with ‘ $+$ ’ sign formally diverges as $m \rightarrow 0$.

Since model (21) with ‘ $-$ ’ sign passes smoothly to the model described by Eq. (11) as $m \rightarrow 0$, it is clear that, for sufficiently small values of m , the evolution described by (21) with ‘ $-$ ’ sign will not be very different from that described by the limiting Eq. (11) with the same initial values of the scale factor and energy density, and the bounce will take place in the case of $\epsilon = -1$. This is explained by the fact that the last term in the second line of (21) has a negative sign and grows by absolute value during the contracting phase. In particular, the bounce will definitely take place in the case of spatially flat or closed universe with zero dark-energy term ($C = 0$) if $\Lambda\ell^2/6 < 1 \Rightarrow \Lambda m^4/M^6 < 3/2$. If the value of m is not so small as to lead to the bounce of the contracting universe, then such a universe will reach a singularity similar to that of the case $\epsilon = 1$ described in our paper [19]. Specifically, as the universe collapses, its energy density ρ grows and can, under certain assumptions, exceed the dark radiation term under the square root in (21). Further increase in ρ will cause the expression under the square root to reach zero, heralding the formation of a cosmological singularity beyond which the solution cannot be continued. This singularity is unusual, since the energy density and pressure remain finite, while the space-time curvature of the brane and its extrinsic curvature tend to infinity as the singularity is approached (see [19] for more details).

Our general conclusions are therefore the following:

1. The four-dimensional cosmological evolution of a Randall–Sundrum-type model with a timelike extra dimension is non-singular both in the past and in the future provided matter

satisfies the weak energy condition $\rho + p \geq 0$. The ‘big crunch/big bang’ singularities are also absent in a broad class of anisotropic Bianchi I braneworld models with a timelike extra dimension.

2. The presence of an ‘induced’ brane curvature term in the higher-dimensional action (1) can trigger the formation of an initial or final singularity even when the extra dimension is timelike. However, the properties of this singularity are unusual since the Hubble parameter remains finite while the curvature tensor diverges as the singularity is approached.

Thus, braneworlds with timelike extra dimensions have properties which are fundamentally different both from standard general-relativistic behaviour and from the properties of braneworld models with spacelike extra dimensions, and which could have interesting cosmological implications. The important issue of the tachyonic nature of Kaluza–Klein gravitational modes in such theories and the related problem of unitarity has been discussed in [3,4] but demands more extensive examination.

Acknowledgments: The authors acknowledge support from the Indo-Ukrainian program of cooperation in science and technology sponsored by the Department of Science and Technology of India and Ministry of Education and Science of Ukraine. The work of Yu. S. was also supported in part by the INTAS grant for project No. 2000-334. The authors acknowledge useful discussions with Katrin Heitmann and Salman Habib.

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